

# Stability for the spherically symmetric Einstein-Vlasov system—a coercivity estimate

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## Abstract

The stability of static solutions of the spherically symmetric, asymptotically flat Einstein-Vlasov system is studied using a Hamiltonian approach based on energy-Casimir functionals. The main result is a coercivity estimate for the quadratic part of the expansion of the natural energy-Casimir functional about an isotropic steady state. The estimate shows in a quantified way that this quadratic part is positive definite on a class of linearly dynamically accessible perturbations, provided the particle distribution of the steady state is a strictly decreasing function of the particle energy and provided the steady state is not too relativistic. This should be an essential step in a fully non-linear stability analysis for the Einstein-Vlasov system. In the present paper it is exploited for obtaining a linearized stability result.

## 1 Introduction

The aim of the present paper is to investigate the stability of spherically symmetric steady states of the Einstein-Vlasov system against spherically

symmetric perturbations. The system describes, in the context of general relativity, the evolution of an ensemble of particles which interact only via gravity. Galaxies or globular clusters, where the stars play the role of the particles, can be modeled as such ensembles, since collisions among stars are sufficiently rare to be neglected. The particle distribution is given by a density function  $f$  on the tangent bundle  $TM$  of the spacetime manifold  $M$ . We assume that all particles have the same rest mass which is normalized to unity. Hence the particle distribution function is supported on the mass shell

$$PM = \{g_{\alpha\beta}p^\alpha p^\beta = -c^2 \text{ and } p^\alpha \text{ is future pointing}\} \subset TM.$$

Here  $g_{\alpha\beta}$  denotes the Lorentz metric on the spacetime manifold  $M$  and  $p^\alpha$  denote the canonical momentum coordinates corresponding to a choice of local coordinates  $x^\alpha$  on  $M$ ; Greek indices always run from 0 to 3, and we have a specific reason for making the dependence on the speed of light  $c$  explicit. We assume that the coordinates are chosen such that

$$ds^2 = c^2 g_{00} dt^2 + g_{ab} dx^a dx^b$$

where Latin indices run from 1 to 3 and  $t = x^0$  should be thought of as a timelike coordinate. On the mass shell,  $p^0$  can be expressed by the remaining coordinates,

$$p^0 = \sqrt{-g^{00}} \sqrt{1 + c^{-2} g_{ab} p^a p^b},$$

and  $f = f(t, x^a, p^b) \geq 0$ . The Einstein-Vlasov system now consists of the Einstein field equations

$$G_{\alpha\beta} = 8\pi c^{-4} T_{\alpha\beta} \tag{1.1}$$

coupled to the Vlasov equation

$$p^0 \partial_t f + p^a \partial_{x^a} f - \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \partial_{p^\alpha} f = 0 \tag{1.2}$$

via the following definition of the energy momentum tensor:

$$T_{\alpha\beta} = c|g|^{1/2} \int p_\alpha p_\beta f \frac{dp^1 dp^2 dp^3}{-p_0}. \tag{1.3}$$

Here  $|g|$  denotes the modulus of the determinant of the metric, and  $\Gamma_{\beta\gamma}^\alpha$  are the Christoffel symbols induced by the metric. We note that the characteristic system of the Vlasov equation (1.2) are the geodesic equations written as a first order system on the mass shell  $PM$  which is invariant under the

geodesic flow. For more background on the Einstein-Vlasov equation we refer to [1].

The system possesses a large collection of static, spherically symmetric solutions obtained for example in [18, 23, 24]. The stability properties of these steady states have essentially not yet been investigated in the mathematics literature; numerical investigations of this question were reported on in [4], and in [25] the author employed variational methods for constructing steady states as minimizers of certain energy-Casimir type functionals. Examples from the astrophysics literature where this stability issue is discussed include [11, 12, 26, 27].

In contrast, for the Vlasov-Poisson system, which arises from the Einstein-Vlasov system in the non-relativistic limit, considerable mathematical progress on the question of the stability of spherically symmetric steady states has been made, cf. [7, 8, 9, 10, 14, 15, 16] and the references in the review articles [17, 20]. For isotropic steady states where the particle distribution  $f_0$  depends only on the local or particle energy the basic stability condition is that this dependence is strictly decreasing on the support of  $f_0$ . In addition to the conserved energy or Hamiltonian the transport structure of the Vlasov equation gives rise to a continuum of conserved quantities, the so-called Casimir functionals. Using the latter the dynamics of the system can in a natural way be restricted to a leaf  $\mathcal{S}_{f_0}$  of perturbations  $g : \mathbb{R}^6 \rightarrow \mathbb{R}$  which have the same level sets as the steady state  $f_0$  under investigation. On this leaf it is then possible to establish a coercivity estimate for the second variation of the Hamiltonian about  $f_0$ . Such an estimate sometimes goes by the name of Antonov's stability bound. Under the assumption of spherical symmetry this estimate was used in [9, 15] to obtain non-linear stability against spherically symmetric perturbations for a large class of steady states  $f_0$ . The case of general perturbations was finally completed in [16].

For the present case of the Einstein-Vlasov system we intend to employ an analogous approach. However, the numerical investigations in [4, 27] indicate an essential difference in the stability behavior between the Vlasov-Poisson and the Einstein-Vlasov systems. For a given microscopic equation of state, i.e., a given dependence of  $f_0$  on the particle energy, there typically exists a one-parameter family of corresponding steady states. For the Vlasov-Poisson system all the members of one such family show the same stability behavior, but this is not so for the Einstein-Vlasov system. This remarkable phenomenon was first conjectured and numerically observed in the physics literature, most notably in the work of Ze'ldovitch [26, 27]. Here, a family of steady states with the same microscopic equation of state is parametrized by the central redshift which is a measure of how close to

Newtonian the steady state is. Provided that the microscopic equation of state is strictly decreasing, the steady states in such a family are stable when they are close to the Newtonian regime, i.e., when the central redshift is small, but they become unstable as this parameter increases beyond a certain threshold. For more precise statements of this behavior and the role of the so-called fractional binding energy in this context we refer to [4] and the original literature [26, 27].

Any stability analysis for the Einstein-Vlasov system must be able to reflect and deal with the above essential differences to the Newtonian case, and the present one does. We prove linear stability of a suitably defined one-parameter family of steady states with small central redshift, cf. Theorem 6.2. The main tool for the result is a quantified coercivity estimate for the second variation of the energy, analogous to the Antonov coercivity bound in the Newtonian case, cf. Theorem 4.2. This is a non-linear estimate that crucially depends both on the symplectic structure of the Einstein-Vlasov system and on properties of the Einstein field equations satisfied by the steady states.

The paper proceeds as follows. In the next section we formulate the system in coordinates which are suitable for the stability analysis and explicitly introduce  $\gamma = 1/c^2$  as a parameter. We state the form of the ADM mass (or energy) and the Casimir functionals in these coordinates, and compute the quadratic term  $D^2\mathcal{H}_C(f_0)$  in the expansion about the steady state  $f_0$  of a suitable energy-Casimir functional which is chosen such that the linear part in the expansion vanishes. In Section 3 we identify a class of linearly dynamically accessible perturbations which form the natural tangent space of the leaf  $\mathcal{S}_{f_0}$  through  $f_0$  which consists of all states which preserve all the Casimir constraints. The main result of our paper is shown in Section 4: We prove that the second variation of the energy-Casimir is positive definite along the linearly dynamically accessible perturbations, provided that the parameter  $\gamma = 1/c^2$  is small enough. As in the case of the Vlasov-Poisson system this positivity result should play an important role in a future, fully non-linear stability analysis. In the present paper we restrict ourselves to drawing a conclusion on linearized stability. In order to derive such a result we analyze in Section 5 the linearized Einstein-Vlasov system. We provide a suitable existence and uniqueness theory for the corresponding initial value problem and show that the class of linearly dynamically accessible perturbations is invariant under the flow of the linearized system and that this flow preserves the quantity  $D^2\mathcal{H}_C(f_0)$ . In Section 6 we finally state and prove our result on linearized stability where we also deal with the fact that for our coercivity estimate the quantity  $\gamma = 1/c^2$ , which in a given set of units

is a specific constant, has to be chosen sufficiently small. This is done by exploiting a scaling symmetry of the problem which relates the Einstein-Vlasov system with unit speed of light  $c = 1$  to the problem where the speed of light assumes a prescribed value  $c$ . In this analysis the smallness requirement for  $\gamma = 1/c^2$  translates into the requirement that the steady states under investigation are close to Newtonian. As explained above, such a sensitivity of the stability properties to being close to Newtonian or very relativistic is to be expected.

Our analysis is restricted to spherically symmetric steady states  $f_0$  and their stability against spherically symmetric perturbations. In particular the latter restriction is undesirable from a physics point of view. To remove it remains an important and highly challenging open problem; the existence of axially symmetric steady states was shown recently in [3].

To conclude this introduction we mention that the global existence result for the spherically symmetric Einstein-Vlasov system with small initial data [21] can be considered as a stability result for the vacuum solution, but the techniques required for the stability analysis of non-trivial steady states are completely different from such small data results. Moreover, the question of weak cosmic censorship and black hole formation for the asymptotically flat Einstein-Vlasov system is studied in [6] and the latter question is addressed by different methods in [2]. A linearized stability analysis for the Vlasov-Poisson system is much easier than for the present case and was performed in [5]. Concerning the Hamiltonian approach for the Einstein-Vlasov system the reference [12] has been a most valuable source of inspiration.

## 2 The spherically symmetric Einstein-Vlasov system and energy-Casimir functionals

We consider a spherically symmetric and asymptotically flat spacetime. In Schwarzschild coordinates the metric takes the form

$$ds^2 = -c^2 e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.1)$$

Here  $t \in \mathbb{R}$  is the time coordinate,  $r \in [0, \infty[$  is the area radius, i.e.,  $4\pi r^2$  is the area of the orbit of the symmetry group  $\text{SO}(3)$  labeled by  $r$ , and the angles  $\theta \in [0, \pi]$  and  $\varphi \in [0, 2\pi]$  parametrize these orbits;  $c$  denotes the speed of light. The spacetime is required to be asymptotically flat with a regular center which corresponds to the boundary conditions

$$\lim_{r \rightarrow \infty} \lambda(t, r) = \lim_{r \rightarrow \infty} \mu(t, r) = 0 = \lambda(t, 0).$$

In order to formulate the Einstein-Vlasov system we write  $x = (x^a) = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ . Let  $p^\alpha$  denote the canonical momentum coordinates corresponding to the spacetime coordinates  $(x^\alpha) = (t, x^1, x^2, x^3)$  and define

$$v^a = p^a + (e^\lambda - 1) \frac{x \cdot p}{r} \frac{x^a}{r}, \text{ where } x \cdot p = \delta_{ab} x^a p^b.$$

This has the advantage that by the mass shell condition

$$p_0 = -e^\mu \sqrt{1 + \gamma |v|^2}, \text{ where } |v|^2 = \delta_{ab} v^a v^b \text{ and } \gamma := \frac{1}{c^2}.$$

We introduce the abbreviation

$$\langle v \rangle := \sqrt{1 + \gamma |v|^2}. \quad (2.2)$$

In the canonical momentum variables the metric would also appear under the square root sign. The Einstein-Vlasov system now takes the following form:

$$\partial_t f + e^{\mu-\lambda} \frac{v}{\langle v \rangle} \cdot \partial_x f - \left( \dot{\lambda} \frac{x \cdot v}{r} + e^{\mu-\lambda} \frac{1}{\gamma} \mu' \langle v \rangle \right) \frac{x}{r} \cdot \partial_v f = 0, \quad (2.3)$$

$$e^{-2\lambda} (2r\lambda' - 1) + 1 = 8\pi\gamma r^2 \rho, \quad (2.4)$$

$$e^{-2\lambda} (2r\mu' + 1) - 1 = 8\pi\gamma^2 r^2 p, \quad (2.5)$$

$$-e^{-\mu-\lambda} \dot{\lambda} = 4\pi\gamma r j, \quad (2.6)$$

$$e^{-2\lambda} \left( \mu'' + (\mu' - \lambda')(\mu' + \frac{1}{r}) \right) - e^{-2\mu} \left( \ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu}) \right) = 4\pi\gamma^2 q, \quad (2.7)$$

where

$$\rho(t, x) = \int f(t, x, v) \langle v \rangle dv, \quad (2.8)$$

$$p(t, x) = \int f(t, x, v) \left( \frac{x \cdot v}{r} \right)^2 \frac{dv}{\langle v \rangle}, \quad (2.9)$$

$$j(t, x) = \int f(t, x, v) \frac{x \cdot v}{r} dv, \quad (2.10)$$

$$q(t, x) = \int f(t, x, v) \left| \frac{x \times v}{r} \right|^2 \frac{dv}{\langle v \rangle}. \quad (2.11)$$

Here  $\partial_x f$  and  $\partial_v f$  denote the gradients of  $f$  with respect to the  $x$ - and  $v$  variable respectively, and  $\dot{\phantom{x}}$  and  $\dot{\phantom{v}}$  denote partial derivatives with respect to

$t$  or  $r$  respectively. Unless explicitly noted otherwise integrals extend over  $\mathbb{R}^3$ . Spherical symmetry means that

$$f(t, x, v) = f(t, Ax, Av), \quad x, v \in \mathbb{R}^3, \quad A \in \text{SO}(3),$$

i.e.,  $f$  is invariant under the canonical action of  $\text{SO}(3)$  on the mass shell. As a consequence the spatial densities defined in (2.8)–(2.11) are actually functions of  $t$  and  $r = |x|$ . We refer to [19] for details on the derivation of this form of the equations. It has been used both in investigations of global existence for small data [21] and of the formation of black holes [2]. The former investigation also provides a local existence and uniqueness theorem for the spherically symmetric Einstein-Vlasov system for smooth, compactly supported and non-negative initial data

$$f|_{t=0} = \overset{\circ}{f} \in C_c^1(\mathbb{R}^6), \quad \overset{\circ}{f} \geq 0, \quad (2.12)$$

which are compatible with (2.4) in the sense explained shortly.

In a stability analysis the following aspect of the choice of coordinates should be noted: In order to compare unperturbed and perturbed quantities one must define some identification of points in the unperturbed with points in the perturbed spacetime where the various quantities are to be evaluated. A priori, no natural such identification exists, and for the Einstein-Vlasov system this question must be faced not only for points on spacetimes but for points on their tangent bundles or corresponding mass shells respectively. Throughout our analysis we identify points which have the same coordinates in the above set-up. Besides the fact that this choice works it is from a physics point of view motivated in [11].

For what follows it is important to note that given a spherically symmetric state  $f \in C_c^1(\mathbb{R}^6)$  we can explicitly and uniquely solve the field equation (2.4) for  $\lambda$  under the boundary condition  $\lambda(0) = 0$ ,

$$e^{-2\lambda(r)} = 1 - \gamma \frac{2m(r)}{r}, \quad (2.13)$$

where

$$m(r) = 4\pi \int_0^r s^2 \rho(s) ds = \int_{|x| \leq r} \int \langle v \rangle f(x, v) dv dx;$$

we will occasionally write  $\lambda_f$ ,  $\rho_f$ ,  $m_f$  to emphasize that these quantities are determined by  $f$ . Clearly, for (2.13) to define  $\lambda$  on all of  $[0, \infty[$  we have to restrict the set of admissible states. We call a state  $f \in C_c^1(\mathbb{R}^6)$  *admissible* iff it is non-negative, spherically symmetric, and

$$\gamma \frac{2m_f(r)}{r} < 1, \quad r \geq 0;$$

the initial data posed in (2.12) must be admissible in this sense.

We now define the ADM mass and the Casimir functionals as functionals on the set of admissible states:

$$\mathcal{H}(f) = \iint \langle v \rangle f \, dv \, dx, \quad (2.14)$$

$$\mathcal{C}(f) = \iint e^{\lambda_f} \chi(f) \, dv \, dx, \quad (2.15)$$

where  $\chi \in C^1(\mathbb{R})$  with  $\chi(0) = 0$ . These quantities are conserved along solutions launched by admissible initial data:

$$\mathcal{H}(f(t)) = \mathcal{H}(\mathring{f}), \quad \mathcal{C}(f(t)) = \mathcal{C}(\mathring{f})$$

as long as the solution exists; for functions  $f = f(t, x, v)$  we write  $f(t) := f(t, \cdot, \cdot)$ , and the analogous notational convention applies to functions of  $t$  and  $x$  or  $t$  and  $r$ .

We need to recall some facts about steady states of the system (2.3)–(2.10), a more detailed discussion including their dependence on  $\gamma$  is given at the beginning of Section 4. For  $\mu = \mu_0(r)$  time-independent and given we define the local or particle energy

$$E = E(x, v) = e^{\mu_0(r)} \langle v \rangle = e^{\mu_0(r)} \sqrt{1 + \gamma |v|^2}. \quad (2.16)$$

The ansatz

$$f_0(x, v) = \phi(E(x, v)) \quad (2.17)$$

with some suitable microscopic equation of state  $\phi = \phi(E)$  then satisfies the time-independent Vlasov equation and reduces the problem of finding stationary solutions to solving the two field equations (2.4), (2.5) where the source terms  $\rho_0$  and  $p_0$  are now functionals of  $\mu = \mu_0$ . A necessary condition for obtaining a steady state with compact support and finite ADM mass is that  $\phi(E) = 0$  for  $E > E_0$  where  $E_0 > 0$  is some cut-off energy. We assume that  $\phi \in C^2(]-\infty, E_0]) \cap C(\mathbb{R})$  is strictly decreasing on  $]-\infty, E_0]$  and such that a corresponding compactly supported steady state  $(f_0, \lambda_0, \mu_0)$  with induced spatial densities  $\rho_0, p_0$  and finite ADM mass exists. Examples of such functions are provided in [18, 23, 24].

We now discuss on a formal level the energy-Casimir approach towards stability for the Einstein-Vlasov system. We consider an energy-Casimir functional

$$\mathcal{H}_C(f) := \mathcal{H}(f) + \mathcal{C}(f), \quad (2.18)$$



and we expand about some given steady state  $f_0$  of the above form:

$$\mathcal{H}_C(f_0 + \delta f) = \mathcal{H}_C(f_0) + D\mathcal{H}_C(f_0)(\delta f) + D^2\mathcal{H}_C(f_0)(\delta f, \delta f) + O((\delta f)^3). \quad (2.19)$$

A rather lengthy and non-trivial formal computation reveals the following. If the function  $\chi$  which generates the Casimir functional (2.15) is chosen such that

$$\chi'(f_0) = \chi'(\phi(E)) = -E, \text{ i.e. } \chi' = -\phi^{-1},$$

then

$$D\mathcal{H}_C(f_0)(\delta f) = 0$$

and

$$\begin{aligned} D^2\mathcal{H}_C(f_0)(\delta f, \delta f) &= \frac{1}{2} \iint \frac{e^{\lambda_0}}{|\phi'(E)|} (\delta f)^2 dv dx \\ &\quad - \frac{1}{2\gamma} \int_0^\infty e^{\mu_0 - \lambda_0} (2r\mu'_0 + 1) (\delta\lambda)^2 dr. \end{aligned} \quad (2.20)$$

Here  $\delta\lambda$  should be expressed in terms of  $\delta f$  through the variation of (2.13), cf. (3.3) below, and it should be observed that on the support of the steady state  $\phi$  is strictly decreasing and in particular one-to-one. Since  $\phi'(E) = 0$  outside the compact support of the steady state  $f_0$ , the first integral makes sense only for perturbations  $\delta f$  which are supported in the support of the steady state. This is automatically the case for the class of linearly dynamically accessible states defined in Section 3. Just as in the case of the Vlasov-Poisson system the central difficulty in the stability analysis arises from the fact that the two terms in (2.20) are of opposite sign. For the Vlasov-Poisson system it has been shown in [13] that  $D^2\mathcal{H}_c(f_0)$  is positive definite on a suitably defined class of linearly dynamically accessible states, and this fact has played a central role in the non-linear stability analysis in [9, 10]. In Section 4 we prove an analogous positivity result for the case of the Einstein-Vlasov system, and we believe that this should be a useful step towards a non-linear stability result. In this context the precise relation of  $D^2\mathcal{H}_c(f_0)$  to the energy-Casimir functional (2.18) is important which is why we have gone into the issue of the expansion (2.19). However, in the present paper we only exploit  $D^2\mathcal{H}_c(f_0)$  and its properties to obtain a linear stability result. For this the relation (2.19) is in principle irrelevant, the important issue being that  $D^2\mathcal{H}_c(f_0)$  is a conserved quantity along solutions of the linearized Einstein-Vlasov system, linearized about the steady state  $f_0$ . This is shown in Section 5. The consequences of our results for non-linear stability are

under investigation, and in this context the expansion (2.19) will have to be made mathematically precise.

To conclude this section we recall the usual Poisson bracket

$$\{f, g\} := \partial_x f \cdot \partial_v g - \partial_v f \cdot \partial_x g \quad (2.21)$$

for two continuously differentiable functions  $f$  and  $g$  of  $x, v \in \mathbb{R}^3$ . We shall repeatedly use the product rule

$$\{f, gh\} = \{f, g\}h + \{f, h\}g \quad (2.22)$$

and the integration by parts formula

$$\iint \{f, g\} dv dx = 0 \quad (2.23)$$

which together with the product rule implies that

$$\iint \{f, g\} h dv dx = - \iint \{f, h\} g dv dx. \quad (2.24)$$

Since throughout, our functions are spherically symmetric it is sometimes convenient to use the following coordinates which are adapted to this symmetry:

$$r = |x|, \quad w = \frac{x \cdot v}{r}, \quad L = |x \times v|^2;$$

a distribution function  $f = f(x, v)$  is spherically symmetric iff by abuse of notation,

$$f = f(r, w, L).$$

It is also worthwhile to note that  $L$  is conserved along the characteristics of the Vlasov equation (2.3), i.e., due to spherical symmetry angular momentum is conserved along particle trajectories. If  $f$  and  $g$  are spherically symmetric and written in terms of  $r, w, L$ , then

$$\{f, g\} = \partial_r f \partial_w g - \partial_w f \partial_r g. \quad (2.25)$$

Notice further that

$$\frac{1}{\gamma} \{f, E\} = e^{\mu_0} \frac{v}{\langle v \rangle} \cdot \partial_x f - \frac{1}{\gamma} e^{\mu_0} \mu'_0 \langle v \rangle \frac{x}{r} \cdot \partial_v f,$$

which should be compared with the Vlasov equation (2.3).

### 3 Dynamically accessible states

Let  $(f_0, \lambda_0, \mu_0)$  with (2.17) be a fixed stationary solution of the spherically symmetric Einstein-Vlasov system whose stability we want to investigate. We call an admissible state  $f$  *non-linearly dynamically accessible from  $f_0$*  iff for all  $\chi \in C^1(\mathbb{R})$  with  $\chi(0) = 0$ ,

$$\mathcal{C}(f) = \mathcal{C}(f_0). \quad (3.1)$$

This property is preserved by the flow of the Einstein-Vlasov system. The aim of the present section is to develop a suitable concept of linearly dynamically accessible states on which the second variation of  $\mathcal{H}_C$  at  $f_0$  is positive definite while the first one vanishes. Moreover, the set of these linearly dynamically accessible states should be invariant under the flow of the linearized Einstein-Vlasov system. Formally, taking the first variation in (3.1), a suitable definition for  $\delta f$  to be linearly dynamically accessible should be that

$$DC(f_0)(\delta f) = \iint e^{\lambda_0} (\chi'(f_0)\delta f + \chi(f_0)\delta\lambda) dv dx = 0 \quad (3.2)$$

for all  $\chi \in C^1(\mathbb{R})$  with  $\chi(0) = 0$ , where

$$\delta\lambda = \gamma e^{2\lambda_0} \frac{4\pi}{r} \int_0^r s^2 \delta\rho(s) ds \quad (3.3)$$

and

$$\delta\rho(r) = \delta\rho(x) = \int \langle v \rangle \delta f(x, v) dv. \quad (3.4)$$

Condition (3.2) is from an analysis point of view not practical to work and prove estimates with. In order to obtain a more explicit condition on  $\delta f$  we first note a simple fact.

**Lemma 3.1** *Under the assumptions made above,*

$$\int \chi(f_0) dv = -\gamma e^{\mu_0} \int \chi'(f_0) \phi'(E) \frac{w^2}{\langle v \rangle} dv,$$

and hence

$$DC(f_0)(\delta f) = \iint e^{\lambda_0} \chi'(f_0) \left[ \delta f - \gamma e^{\mu_0} \delta\lambda \phi'(E) \frac{w^2}{\langle v \rangle} \right] dv dx.$$

**Proof.** We use the expressions  $E = e^{\mu_0} \langle v \rangle$  and  $L = |x \times v|^2$  as new integration variables and observe that the integrand is even in  $v$ . Hence

$$\begin{aligned} dv &= \frac{2\pi e^{-\mu_0}}{\sqrt{\gamma} r^2} \frac{E}{\sqrt{E^2 - e^{2\mu_0} (1 + \gamma L/r^2)}} dE dL \\ &= \frac{2\pi e^{-\mu_0}}{\sqrt{\gamma} r^2} \partial_E \sqrt{E^2 - e^{2\mu_0} (1 + \gamma L/r^2)} dE dL. \end{aligned}$$

An integration by parts now gives the result:

$$\begin{aligned} &\int \chi(f_0) dv \\ &= \frac{2\pi e^{-\mu_0}}{\sqrt{\gamma} r^2} \int_0^\infty \int_{e^{\mu_0} \sqrt{1 + \gamma \frac{L}{r^2}}}^\infty \chi(\phi(E)) \partial_E \sqrt{E^2 - e^{2\mu_0} \left(1 + \gamma \frac{L}{r^2}\right)} dE dL \\ &= -\frac{2\pi e^{-\mu_0}}{\sqrt{\gamma} r^2} \int_0^\infty \int_{e^{\mu_0} \sqrt{1 + \gamma \frac{L}{r^2}}}^\infty \chi'(\phi(E)) \phi'(E) \sqrt{E^2 - e^{2\mu_0} \left(1 + \gamma \frac{L}{r^2}\right)} dE dL \\ &= -\int \chi'(\phi(E)) \phi'(E) \frac{E^2 - e^{2\mu_0} (1 + \gamma L/r^2)}{E} dv \\ &= -\gamma e^{\mu_0} \int \chi'(f_0) \phi'(E) \frac{w^2}{\langle v \rangle} dv. \end{aligned}$$

□

Using this result we find that a variation  $\delta f$  satisfies the condition (3.2), if

$$e^{\lambda_0} \delta f - \gamma e^{\mu_0 + \lambda_0} \delta \lambda \phi'(E) \frac{w^2}{\langle v \rangle} = \{h, f_0\} \quad (3.5)$$

for some spherically symmetric generating function  $h \in C^2(\mathbb{R}^6)$ . This is due to the fact that

$$\iint \chi'(f_0) \{h, f_0\} dv dx = 0$$

for any such generating function  $h$ . Since  $\delta \lambda$  appears in the second term, (3.5) is still not suitable as a definition for  $\delta f$ , but we have the following result:

**Proposition 3.2** *Let*

$$\delta f := e^{-\lambda_0} \{h, f_0\} + 4\pi \gamma^3 r e^{2\mu_0 + \lambda_0} \phi'(E) \frac{w^2}{\langle v \rangle} \int \phi'(E(x, \tilde{v})) h(x, \tilde{v}) \tilde{w} d\tilde{v} \quad (3.6)$$

for some spherically symmetric generating function  $h \in C^2(\mathbb{R}^6)$ . If the corresponding variation of  $\lambda$  is defined by (3.3), then

$$\delta\lambda = 4\pi r\gamma^2 e^{\mu_0+\lambda_0} \int \phi'(E) h(x, v) w dv. \quad (3.7)$$

Hence  $\delta f$  satisfies both (3.5) and (3.2). States of the form (3.6) are called linearly dynamically accessible from  $f_0$ .

For the proof the following auxiliary result is needed which will also be used elsewhere.

**Lemma 3.3** *The following identity holds:*

$$\int \phi'(E) w^2 dv = -\frac{e^{-\mu_0}}{\gamma} (\gamma p_0 + \rho_0) = -\frac{e^{-2\lambda_0-\mu_0}}{4\pi\gamma^2 r} (\lambda'_0 + \mu'_0).$$

**Proof.** We note that

$$\partial_v \phi(E) = \phi'(E) e^{\mu_0} \frac{\gamma v}{\langle v \rangle}$$

and hence

$$\frac{x}{r} \cdot \partial_v \phi(E) = \phi'(E) e^{\mu_0} \frac{\gamma w}{\langle v \rangle}.$$

This implies that

$$\begin{aligned} \int \phi'(E) w^2 dv &= \frac{e^{-\mu_0}}{\gamma} \int \frac{x}{r} \cdot \partial_v \phi(E) \langle v \rangle w dv \\ &= -\frac{e^{-\mu_0}}{\gamma} \int \phi(E) \left( \frac{\gamma w^2}{\langle v \rangle} + \langle v \rangle \right) dv \\ &= -\frac{e^{-\mu_0}}{\gamma} (\gamma p_0 + \rho_0) = -\frac{e^{-2\lambda_0-\mu_0}}{4\pi\gamma^2 r} (\lambda'_0 + \mu'_0); \end{aligned}$$

the last equality is due to the field equations (2.4), (2.5) for the steady state.  $\square$

**Proof of Proposition 3.2.** The variation of  $\rho$  induced by  $\delta f$  takes the form

$$\delta\rho = e^{-\lambda_0} \int \langle v \rangle \{h, f_0\} dv + 4\pi r\gamma^3 e^{\lambda_0+2\mu_0} \int \phi'(E) h w dv \int \phi'(E) w^2 dv.$$

Hence Lemma 3.3 implies that

$$\delta\rho = e^{-\lambda_0} \int \langle v \rangle \{h, f_0\} dv - \gamma e^{\mu_0-\lambda_0} (\lambda'_0 + \mu'_0) \int \phi'(E) h w dv.$$

We consider the first term on the right hand side and find that

$$\begin{aligned}
& \int \langle v \rangle \{h, f_0\} dv \\
&= \int \langle v \rangle \left( \partial_x h \cdot \frac{e^{\mu_0} \gamma v}{\langle v \rangle} - \partial_v h \cdot \frac{x}{r} \mu'_0 e^{\mu_0} \langle v \rangle \right) \phi'(E) dv \\
&= \gamma e^{\mu_0} \int \partial_x h \cdot v \phi'(E) dv - \mu'_0 e^{\mu_0} \int \partial_v h \cdot \frac{x}{r} (1 + \gamma |v|^2) \phi'(E) dv \\
&= \gamma e^{\mu_0} \int \left[ \partial_x h \cdot v \phi'(E) + 2\mu'_0 w h \phi'(E) + e^{\mu_0} \mu'_0 w h \phi''(E) \langle v \rangle \right] dv.
\end{aligned}$$

Hence

$$\begin{aligned}
\delta \lambda &= e^{2\lambda_0} \frac{\gamma^2}{r} \int_{|x| \leq r} \int e^{\mu_0 - \lambda_0} \left[ \partial_x h \cdot v \phi'(E) + 2\mu'_0 w h \phi'(E) \right. \\
&\quad \left. + e^{\mu_0} \mu'_0 w h \phi''(E) \langle v \rangle - (\lambda'_0 + \mu'_0) \phi'(E) h w \right] dv \\
&= e^{2\lambda_0} \frac{\gamma^2}{r} 4\pi r^2 e^{\mu_0 - \lambda_0} \int w h \phi'(E) dv \\
&\quad + e^{2\lambda_0} \frac{\gamma^2}{r} \int_{|x| \leq r} \int e^{\mu_0 - \lambda_0} \left[ -(\mu'_0 - \lambda'_0) w \phi'(E) - \phi''(E) e^{\mu_0} \mu'_0 w \langle v \rangle \right. \\
&\quad \left. + 2\mu'_0 w \phi'(E) + \mu'_0 e^{\mu_0} w \langle v \rangle \phi''(E) \right. \\
&\quad \left. - (\lambda'_0 + \mu'_0) \phi'(E) w \right] h dv dx,
\end{aligned}$$

and since the term in brackets vanishes the claim is proven.  $\square$

**Remark.** For linearly dynamically accessible states as defined in Proposition 3.2,

$$\begin{aligned}
\delta f &= \phi'(E) \left( e^{-\lambda_0} \{h, E\} + 4\pi \gamma^3 r e^{2\mu_0 + \lambda_0} \frac{w^2}{\langle v \rangle} \int \phi'(E) h(x, v) w dv \right) \\
&= \phi'(E) \left( e^{-\lambda_0} \{h, E\} + \gamma r e^{\mu_0} \delta \lambda \frac{w^2}{\langle v \rangle} \right),
\end{aligned}$$

in particular,  $\delta f$  vanishes outside the support of  $f_0$  and the integrals in (2.20) are well defined.

## 4 The coercivity estimate

The central step in our stability analysis is a coercivity estimate for the second variation of the energy. We recall the definition of this expression

which is also referred to as the free energy:

$$\begin{aligned}\mathcal{A}(\delta f) &:= D^2\mathcal{H}_C(f_0)(\delta f, \delta f) \\ &= \frac{1}{2} \iint \frac{e^{\lambda_0}}{|\phi'(E)|} (\delta f)^2 dv dx - \frac{1}{2\gamma} \int_0^\infty e^{\mu_0 - \lambda_0} (2r\mu'_0 + 1) (\delta\lambda)^2 dr.\end{aligned}\tag{4.1}$$

The estimate will hold provided  $\gamma$  is sufficiently small. We need to make precise the assumptions on the steady states under consideration and need to discuss their behavior for  $\gamma \rightarrow 0$ , i.e., their relation to those of the Vlasov-Poisson system

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0, \tag{4.2}$$

$$\Delta U = 4\pi\rho, \quad \lim_{|x| \rightarrow \infty} U(t, x) = 0, \tag{4.3}$$

$$\rho(t, x) = \int f(t, x, v) dv. \tag{4.4}$$

Here  $U = U(t, x)$  is the gravitational potential of the ensemble, and  $\rho = \rho(t, x)$  is its spatial mass density. The Newtonian particle energy is given by

$$E = E(x, v) = \frac{1}{2}|v|^2 + U(x), \tag{4.5}$$

and under suitable assumptions on  $\Phi$  the ansatz

$$f_0(x, v) = \Phi\left(\frac{1}{2}|v|^2 + U(x)\right) \tag{4.6}$$

leads to spherically symmetric steady states.

**Assumption on  $\Phi$ .** Let  $\Phi \in C(\mathbb{R}) \cap C^2(]-\infty, 0[)$  with

$$\Phi = 0 \text{ on } [0, \infty[ \text{ and } \Phi' < 0 \text{ on } ]-\infty, 0[$$

be such that for a parameter  $\mathring{\nu} < 0$  the semilinear Poisson equation

$$\frac{1}{r^2}(r^2 U'(r))' = 4\pi \int_{U(r)}^\infty \Phi(E) \sqrt{2(E - U(r))} dE \tag{4.7}$$

has a unique solution  $U \in C^2([0, \infty[)$  with the central value  $U(0) = \mathring{\nu}$  and the property that  $U(R_0) = 0$  for some radius  $R_0 > 0$ .

The ansatz (4.6) necessarily leads to a spherically symmetric steady state where  $U = U(r)$ ,  $r = |x|$ , and reduces the static Vlasov-Poisson system to the equation (4.7); the corresponding steady state is supported in space in the ball of radius  $R_0$  about the origin. We also remark that the boundary

condition for the potential at spatial infinity has been replaced by the condition that the potential vanishes at the boundary of the support of the steady state. This is technically advantageous, and the original boundary condition can be restored by a suitable shift of the potential and a corresponding cut-off energy  $E_0 < 0$ . A well known example where our assumptions hold are the polytropic steady states given by

$$\Phi(E) = \begin{cases} (-E)^k, & E < 0, \\ 0, & E \geq 0 \end{cases}$$

with  $k \in ]0, 7/2[$ ; more examples can be found in [24].

For the Einstein-Vlasov system the particle energy is given by

$$E = E(x, v) = e^{\mu_0(r)} \langle v \rangle = e^{\mu_0(r)} \sqrt{1 + \gamma |v|^2}.$$

We fix a function  $\Phi$  as above and write  $\mu_0 = \gamma \nu_0$ . By the ansatz

$$f_0(x, v) = \Phi \left( \frac{1}{\gamma} \sqrt{1 + \gamma |v|^2} e^{\gamma \nu_0} - \frac{1}{\gamma} \right) = \Phi \left( \frac{E - 1}{\gamma} \right)$$

the static Einstein-Vlasov system is reduced to a single equation for  $\nu_0$ , namely

$$\nu'_0(r) = \frac{4\pi}{1 - \frac{8\pi}{r} \gamma \int_0^r s^2 g_\gamma(\nu_0(s)) ds} \left( \gamma r h_\gamma(\nu_0(r)) + \frac{1}{r^2} \int_0^r s^2 g_\gamma(\nu_0(s)) ds \right), \quad (4.8)$$

where  $g_\gamma$  and  $h_\gamma$  are smooth functions determined by  $\Phi$  which are such that  $\rho_0(r) = g_\gamma(\nu_0(r))$  and  $p_0(r) = h_\gamma(\nu_0(r))$ . The important point is that

$$g_\gamma(\nu_0(r)) \rightarrow \int_{\nu_0(r)}^\infty \Phi(E) \sqrt{2(E - \nu_0(r))} dE \text{ as } \gamma \rightarrow 0. \quad (4.9)$$

It should be noticed that the ansatz (2.17) has been adapted to yield the proper Newtonian limit, and we have the relation  $\phi(\cdot) = \Phi(\frac{\cdot}{\gamma} - \frac{1}{\gamma})$ . For the details of these arguments we have to refer to [23]; here we collect only the information which we need for the stability analysis.

**Proposition 4.1** *There exist constants  $\gamma_0 > 0$  and  $C > 0$  such that for  $0 < \gamma \leq \gamma_0$  the equation (4.8) has a unique solution  $\nu_0 \in C^2([0, \infty[)$  with  $\nu_0(0) = \hat{\nu}$ . The resulting steady state  $(f_0, \lambda_0, \mu_0 = \gamma \nu_0)$  satisfies the following estimates:*

$$|x| + |v| \leq C \text{ and } \nu'_0 > \frac{1}{C} \text{ on } \text{supp} f_0,$$

and

$$\|\rho_0\|_\infty, \|p_0\|_\infty, \|\lambda_0\|_\infty, \|\nu_0\|_\infty, \|\nu'_0\|_\infty \leq C.$$



**Sketch of the proof.** By the assumption on  $\Phi$  the Newtonian potential  $U$  has a zero at some radius  $R_0$ , and since  $U$  is strictly increasing it is strictly positive for  $r > R_0$ . The structure of the right hand side of (4.8) and in particular the limiting behavior (4.9) implies that  $\nu_0$  converges to  $U$  uniformly on bounded intervals as  $\gamma \rightarrow 0$ , in particular,  $\nu_0$  must also have a zero at a radius close to  $R_0$  for  $\gamma$  small, and the rest follows; for details we refer to [23].  $\square$

The content of the following theorem is the coercivity estimate for the free energy  $\mathcal{A}$  on linearly dynamically accessible perturbations.

**Theorem 4.2** *There exist constants  $C^* > 0$  and  $\gamma^* > 0$  such that for any  $0 < \gamma \leq \gamma^*$  and any spherically symmetric function  $h \in C^2(\mathbb{R}^6)$  which is odd in the  $v$ -variable the estimate*

$$\mathcal{A}(\delta f) \geq C^* \iint |\phi'(E)| \left( (rw)^2 \left| \left\{ E, \frac{h}{rw} \right\} \right|^2 + \gamma^2 |h|^2 \right) dv dx$$

*holds. Here  $\delta f$  is the dynamically accessible perturbation generated by  $h$  according to (3.6).*

Before proving this theorem we collect some auxiliary results.

**Lemma 4.3** *Let  $h \in C^2(\mathbb{R}^6)$  be spherically symmetric. Then the following estimate holds:*

$$\left( \int |\phi'(E)| wh dv \right)^2 \leq \frac{e^{-2\lambda_0 - \mu_0}}{4\pi\gamma^2 r} (\lambda'_0 + \mu'_0) \int |\phi'(E)| h^2 dv.$$

**Proof.** By the Cauchy-Schwarz inequality,

$$\left( \int |\phi'(E)| wh dv \right)^2 \leq \int |\phi'(E)| w^2 dv \int |\phi'(E)| h^2 dv,$$

and applying Lemma 3.3 completes the proof.  $\square$

**Lemma 4.4** *The following identities hold:*

$$\begin{aligned} \{E, rw\} &= e^{\mu_0} r \mu'_0 \langle v \rangle - e^{\mu_0} \langle v \rangle + \frac{e^{\mu_0}}{\langle v \rangle}, \\ \{E, \{E, rw\}\} &= -\gamma e^{2\mu_0} w \left[ r \mu''_0 + \mu'_0 + \frac{2\mu'_0}{\langle v \rangle^2} \right]. \end{aligned}$$

**Proof.** The first identity follows easily from the product rule (2.22) and the definition (2.16) of  $E$ , for the second identity we use the first one and again the product rule.  $\square$

**Proof of Theorem 4.2.** Before starting our estimates we mention that throughout the following proof,  $0 < \gamma \leq \gamma_0$  as in Proposition 4.1, and  $C$  denotes a positive constant which may change its value from line to line, depends on the bounds provided in Proposition 4.1, and is independent of  $\gamma$ .

*Step 1: Splitting  $\mathcal{A}$ .* Using the formula (3.6) for  $\delta f$  and the definition (4.1) of  $\mathcal{A}$ , we expand the square of  $\delta f$  appearing under the integral sign and rewrite  $\mathcal{A}(\delta f, \delta \lambda)$  in the following form:

$$2\mathcal{A}(\delta f) = \mathcal{A}_1(\delta f) + \mathcal{A}_2(\delta f), \quad (4.10)$$

where

$$\begin{aligned} \mathcal{A}_1(\delta f) &:= \iint e^{-\lambda_0} |\phi'(E)| |\{E, h\}|^2 dv dx \\ &\quad - \frac{1}{\gamma} \int_0^\infty e^{\mu_0 - \lambda_0} (2r\mu'_0 + 1) (\delta \lambda)^2 dr, \\ &=: \mathcal{A}_{11}(\delta f) + \mathcal{A}_{12}(\delta f), \end{aligned} \quad (4.11)$$

$$\begin{aligned} \mathcal{A}_2(\delta f) &:= -2\gamma \iint |\phi'(E)| \{E, h\} \delta \lambda e^{\mu_0} \frac{w^2}{\langle v \rangle} dv dx \\ &\quad + \gamma^2 \iint |\phi'(E)| e^{2\mu_0 + \lambda_0} \frac{w^4}{\langle v \rangle^2} (\delta \lambda)^2 dv dx \\ &=: \mathcal{A}_{21}(\delta f) + \mathcal{A}_{22}(\delta f). \end{aligned} \quad (4.12)$$

The idea behind this splitting is that  $\mathcal{A}_1$  will yield the desired lower bound while  $\mathcal{A}_2$  is of higher order in  $\gamma$  and can eventually be controlled by the positive contribution from  $\mathcal{A}_1$ .

*Step 2: The term  $\mathcal{A}_1$ .* Let us define

$$\eta := \frac{1}{rw} h.$$

The function  $\eta$  is well-defined for  $w = 0$  since  $h$  is odd in  $v$ . By the product rule (2.22),

$$\{E, h\} = rw\{E, \eta\} + \eta\{E, rw\}, \quad (4.13)$$

and by (4.13),

$$|\{E, h\}|^2 = (rw)^2 |\{E, \eta\}|^2 + \{E, \eta^2 rw\{E, rw\}\} - \eta^2 rw\{E, \{E, rw\}\}.$$

Inserting this into  $\mathcal{A}_{11}$ , we arrive at

$$\begin{aligned}
\iint e^{-\lambda_0} |\phi'(E)| |\{E, h\}|^2 dv dx &= \iint e^{-\lambda_0} |\phi'(E)| (rw)^2 |\{E, \eta\}|^2 dv dx \\
&+ \iint e^{-\lambda_0} |\phi'(E)| \{E, \eta^2 rw \{E, rw\}\} dv dx \\
&- \iint e^{-\lambda_0} |\phi'(E)| \eta^2 rw \{E, \{E, rw\}\} dv dx \\
&=: U + V + W.
\end{aligned} \tag{4.14}$$

The term  $U$  is a positive definite contribution to  $\mathcal{A}_1$ . The terms  $V$  and  $W$  are more delicate since they will have to be combined with the negative term  $\mathcal{A}_{12}$ . For the term  $V$  we use the formula (2.24) and integrate by parts in  $v$  to arrive at

$$\begin{aligned}
V &= - \iint e^{-\lambda_0} \{f_0, \eta^2 rw \{E, rw\}\} dv dx \\
&= \iint f_0 \{e^{-\lambda_0}, \eta^2 rw \{E, rw\}\} dv dx \\
&= \iint f_0 \partial_x (e^{-\lambda_0}) \cdot \partial_v (\eta^2 rw \{E, rw\}) dv dx \\
&= - \iint \partial_v f_0 \cdot \frac{x}{r} e^{-\lambda_0} (-\lambda'_0) \eta^2 rw \{E, rw\} dv dx \\
&= \iint \phi'(E) e^{\mu_0} \frac{\gamma v}{\langle v \rangle} \cdot \frac{x}{r} e^{-\lambda_0} \lambda'_0 \eta^2 rw \{E, rw\} dv dx.
\end{aligned}$$

Using the first identity in Lemma 4.4,

$$\begin{aligned}
V &= -\gamma \iint |\phi'(E)| e^{2\mu_0 - \lambda_0} \lambda'_0 \frac{\eta^2 rw^2}{\langle v \rangle} \left( r \mu'_0 \langle v \rangle - \langle v \rangle + \frac{1}{\langle v \rangle} \right) dv dx \\
&= -\gamma \iint |\phi'(E)| e^{2\mu_0 - \lambda_0} h^2 \left( \lambda'_0 \mu'_0 - \frac{\lambda'_0}{r} + \frac{\lambda'_0}{r \langle v \rangle^2} \right) dv dx.
\end{aligned}$$

By the second identity in Lemma 4.4,

$$W = \gamma \iint |\phi'(E)| e^{2\mu_0 - \lambda_0} h^2 \left( \mu''_0 + \frac{\mu'_0}{r} + \frac{2\mu'_0}{r \langle v \rangle^2} \right) dv dx,$$

and hence

$$V + W = \gamma \iint |\phi'(E)| e^{2\mu_0 - \lambda_0} h^2 \left( \mu''_0 - \lambda'_0 \mu'_0 + \frac{\mu'_0 + \lambda'_0}{r} + \frac{2\mu'_0 - \lambda'_0}{r \langle v \rangle^2} \right) dv dx. \tag{4.15}$$

In order to bound the expression  $\mathcal{A}_{12}$  from below, we use the formula (3.7) for  $\delta\lambda$  and Lemma 4.3:

$$\begin{aligned}
& -\frac{1}{\gamma} \int_0^\infty e^{\mu_0 - \lambda_0} (2r\mu'_0 + 1) (\delta\lambda)^2 dr \\
& = -\frac{1}{\gamma} \int_0^\infty e^{\mu_0 - \lambda_0} (2r\mu'_0 + 1) 16\pi^2 r^2 \gamma^4 e^{2\mu_0 + 2\lambda_0} \left( \int \phi'(E) h w dv \right)^2 dr \\
& \geq -4\pi\gamma^3 \int e^{3\mu_0 + \lambda_0} (2r\mu'_0 + 1) \frac{e^{-2\lambda_0 - \mu_0}}{4\pi\gamma^2 r} (\lambda'_0 + \mu'_0) \int |\phi'(E)| h^2 dv dx \\
& = -\gamma \iint |\phi'(E)| e^{2\mu_0 - \lambda_0} h^2 \left( 2\mu'_0 \lambda'_0 + 2(\mu'_0)^2 + \frac{\mu'_0 + \lambda'_0}{r} \right) dv dx. \quad (4.16)
\end{aligned}$$

From (4.15) and (4.16) we have

$$\begin{aligned}
V + W & - \frac{1}{\gamma} \int_0^\infty e^{\mu_0 - \lambda_0} (2r\mu'_0 + 1) (\delta\lambda)^2 dr \\
& \geq \gamma \iint |\phi'(E)| e^{2\mu_0 - \lambda_0} h^2 \left[ \mu''_0 - 3\mu'_0 \lambda'_0 - 2(\mu'_0)^2 + \frac{2\mu'_0 - \lambda'_0}{r \langle v \rangle^2} \right]. \quad (4.17)
\end{aligned}$$

In order to estimate the term in the rectangular brackets on the right-hand side of (4.17), we express  $\mu''_0$  via the field equation (2.7) for the steady state and obtain, using the fact that the source term  $q_0$  in the latter equation is non-negative,

$$\begin{aligned}
[\dots] & = 4\pi\gamma^2 q_0 e^{2\lambda_0} - 2\mu'_0 \lambda'_0 - 3(\mu'_0)^2 - \frac{\mu'_0}{r} + \frac{\lambda'_0}{r} + 2\frac{\mu'_0}{r \langle v \rangle^2} - \frac{\lambda'_0}{r \langle v \rangle^2} \\
& \geq -2\mu'_0 (\mu'_0 + \lambda'_0) - (\mu'_0)^2 + \frac{\mu'_0}{r} \left( \frac{2}{\langle v \rangle^2} - 1 \right) + \frac{\lambda'_0}{r} \left( 1 - \frac{1}{\langle v \rangle^2} \right) \\
& = -8\pi\gamma^2 r e^{2\lambda_0} \nu'_0 (\rho_0 + \gamma p_0) - \gamma^2 (\nu'_0)^2 + \frac{\gamma \nu'_0}{r} \frac{1 - \gamma |v|^2}{\langle v \rangle^2} + \frac{\lambda'_0}{r} \frac{\gamma |v|^2}{\langle v \rangle^2}.
\end{aligned}$$

The observation that

$$\lambda'_0 = \lambda'_0 + \mu'_0 - \mu'_0 = \gamma \left( 4\pi r (\rho_0 + \gamma p_0) e^{2\lambda_0} - \nu'_0 \right) \geq -\gamma \nu'_0$$

implies that

$$\begin{aligned}
[\dots] & \geq \gamma \frac{\nu'_0}{r \langle v \rangle^2} - \gamma^2 \left( 8\pi r \nu'_0 e^{2\lambda_0} (\rho_0 + \gamma p_0) + (\nu'_0)^2 + 2\frac{\nu'_0 |v|^2}{r \langle v \rangle^2} \right) \\
& = \gamma \frac{\nu'_0}{r \langle v \rangle^2} \left[ 1 - \gamma \left( 8\pi r^2 \langle v \rangle^2 e^{2\lambda_0} (\rho_0 + \gamma p_0) + r \langle v \rangle^2 \nu'_0 + 2|v|^2 \right) \right].
\end{aligned}$$

Using Proposition 4.1 there exists a constant  $\gamma_1 \in ]0, \gamma_0]$  such that for all  $0 < \gamma \leq \gamma_1$ ,

$$[\dots] \geq \frac{\gamma}{4} \frac{\nu'_0}{r}. \quad (4.18)$$

Therefore, from (4.14) and (4.16), assuming this smallness condition for  $\gamma$  we obtain the crucial estimate

$$\begin{aligned} \mathcal{A}_1(\delta f) &\geq \iint |\phi'(E)| \left( e^{-\lambda_0} (rw)^2 |\{E, \eta\}|^2 + \frac{\gamma^2}{4} e^{2\mu_0 - \lambda_0} \frac{\nu'_0}{r} h^2 \right) dv dx \\ &\geq C \iint |\phi'(E)| ((rw)^2 |\{E, \eta\}|^2 + \gamma^2 h^2) dv dx, \end{aligned} \quad (4.19)$$

where we again used the bounds from Proposition 4.1.

*Step 3: The term  $\mathcal{A}_2$ .* Since this term is of higher order in  $\gamma$ , we expect to be able to prove that it is small compared to  $\mathcal{A}_1$  for suitably small  $\gamma$ . Using the decomposition (4.13) and keeping in mind that  $\eta = h/rw$  and the formula (3.7) for  $\delta\lambda$  we can rewrite the part  $\mathcal{A}_{21}$  as follows:

$$\begin{aligned} \mathcal{A}_{21} &= -8\pi\gamma^3 \iint |\phi'(E)| e^{2\mu_0 + \lambda_0} \frac{rw^2}{\langle v \rangle} \{E, h\} \left( \int \phi'(E) h \tilde{w} d\tilde{v} \right) dv dx \\ &= -8\pi\gamma^3 \iint |\phi'(E)| e^{2\mu_0 + \lambda_0} \frac{r^2 w^3}{\langle v \rangle} \{E, \eta\} \left( \int \phi'(E) h \tilde{w} d\tilde{v} \right) dv dx \\ &\quad - 8\pi\gamma^3 \iint |\phi'(E)| e^{2\mu_0 + \lambda_0} \frac{w}{\langle v \rangle} \{E, rw\} h \left( \int \phi'(E) h \tilde{w} d\tilde{v} \right) dv dx \\ &=: X + Y. \end{aligned}$$

By Lemma 4.3,

$$\left( \int |\phi'(E)| w h dv \right)^2 \leq \frac{e^{-\mu_0}}{\gamma} (\rho_0 + \gamma p_0) \int |\phi'(E)| h^2 dv \leq \frac{C}{\gamma} \int |\phi'(E)| h^2 dv. \quad (4.20)$$

In addition, Lemma 3.3 together with the bounds from Proposition 4.1 imply that

$$\sup_{x \in \mathbb{R}^3} \int |\phi'(E)| w^2 dv \leq \frac{C}{\gamma}. \quad (4.21)$$

The bounds from Proposition 4.1, the Cauchy-Schwarz inequality, and the estimates (4.20), (4.21) imply that

$$\begin{aligned} |X| &\leq C\gamma^3 \int \left| \int |\phi'(E)|^{1/2} w |\phi'(E)|^{1/2} rw \{E, \eta\} dv \right| \left| \int |\phi'(E)| h w dv \right| dx \\ &\leq C\gamma^{5/2} \left( \iint |\phi'(E)| |rw \{E, \eta\}|^2 dv dx \right)^{1/2} \left( \frac{1}{\gamma} \iint |\phi'(E)| h^2 dv dx \right)^{1/2}. \end{aligned}$$

Hence by (4.19),

$$|X| \leq C\gamma\mathcal{A}_1.$$

In order to estimate the term  $Y$  we rewrite the first identity of Lemma 4.4:

$$\{E, rw\} = e^{\mu_0} r\gamma\nu'_0 \langle v \rangle + e^{\mu_0} \left( -\langle v \rangle + \frac{1}{\langle v \rangle} \right) = \gamma e^{\mu_0} \left( r\nu'_0 - \frac{v^2}{\langle v \rangle} \right).$$

Thus,

$$\sup_{\text{supp } f_0} |\{E, rw\}| \leq C\gamma.$$

Using this together with the estimates from Proposition 4.1 we proceed as above to find that

$$\begin{aligned} |Y| &\leq C\gamma^{7/2} \left( \iint |\phi'(E)| h^2 dv dx \right)^{1/2} \left( \frac{1}{\gamma} \iint |\phi'(E)| h^2 dv dx \right)^{1/2} \\ &\leq C\gamma\mathcal{A}_1. \end{aligned}$$

From the above estimates for  $|X|$  and  $|Y|$  it follows that

$$\mathcal{A}_2 \geq -|X| - |Y| \geq -C\gamma\mathcal{A}_1.$$

From this estimate and (4.10), we finally infer that

$$\mathcal{A} \geq \frac{1}{2}\mathcal{A}_1 - C\gamma\mathcal{A}_1 \geq \frac{1}{4}\mathcal{A}_1$$

provided  $\gamma$  is sufficiently small. In view of (4.19) the proof is complete.  $\square$

**Remark.** The estimate (4.19) shows that an equivalent way of stating our coercivity estimate is

$$\mathcal{A}(\delta f) \geq \frac{1}{2} \iint |\phi'(E)| \left( e^{-\lambda_0} (rw)^2 |\{E, \eta\}|^2 + \frac{\gamma^2}{4} e^{2\mu_0 - \lambda_0} \frac{\nu'_0}{r} h^2 \right) dv dx,$$

which should be compared with the coercivity estimate in the Newtonian case, cf. [9, Lemma 1.1].

If we want to use Theorem 4.2 to deduce a stability result the restriction that  $h$  is odd in  $v$  is a problem because the generating functions of our perturbations need not be odd and more importantly, even if they were this property is not preserved under the linearized flow. However, the restriction is easily removed. To this end, we define for a function  $h = h(x, v)$  its even and odd parts with respect to  $v$  as usual by

$$h_+(x, v) := \frac{1}{2} (h(x, v) + h(x, -v)), \quad h_-(x, v) := \frac{1}{2} (h(x, v) - h(x, -v)).$$

**Corollary 4.5** *For any spherically symmetric function  $h \in C^2(\mathbb{R}^6)$  the estimate*

$$\begin{aligned} \mathcal{A}(\delta f) &\geq C^* \iint |\phi'(E)| \left( (rw)^2 \left| \left\{ E, \frac{h_-}{rw} \right\} \right|^2 + \gamma^2 |h_-|^2 \right) dv dx \\ &\quad + \frac{1}{2} \iint e^{-\lambda_0} |\phi'(E)| |\{E, h_+\}|^2 dv dx \end{aligned}$$

*holds. Here  $\delta f$  is the dynamically accessible perturbation generated by  $h$ ,  $C^* > 0$  and  $\gamma^* > 0$  are as in Theorem 4.2, and  $0 < \gamma \leq \gamma^*$ .*

**Proof.** We split  $h$  into its even and odd parts,  $h = h_+ + h_-$ . Then

$$\delta \lambda = 4\pi r \gamma^2 e^{\mu_0 + \lambda_0} \int \phi'(E) h_-(x, v) w dv,$$

since  $h_+$  does not contribute to the last integral. Since  $f_0$  is even in  $v$  this implies that

$$\begin{aligned} \delta f_- &= e^{-\lambda_0} \{h_+, f_0\}, \\ \delta f_+ &= e^{-\lambda_0} \{h_-, f_0\} + \gamma e^{\mu_0} \phi'(E) \frac{w^2}{\langle v \rangle} \delta \lambda, \end{aligned}$$

in particular, the even part of  $\delta f$  is the dynamically accessible perturbation induced by the odd part of  $h$ . Hence

$$\begin{aligned} \mathcal{A}(\delta f) &= \mathcal{A}(\delta f_+) + \iint e^{\lambda_0} \frac{\delta f_+ \delta f_-}{|\phi'(E)|} dv dx + \frac{1}{2} \iint e^{\lambda_0} \frac{|\delta f_-|^2}{|\phi'(E)|} dv dx \\ &= \mathcal{A}(\delta f_+) + \frac{1}{2} \iint e^{-\lambda_0} |\phi'(E)| |\{E, h_+\}|^2 dv dx, \end{aligned}$$

since the integrand of the mixed term is odd in  $v$ . The assertion follows if we now apply Theorem 4.2.  $\square$

## 5 The linearized Einstein-Vlasov system

In order to linearize the spherically symmetric Einstein-Vlasov system (2.3)–(2.10) about a given steady state  $(f_0, \lambda_0, \mu_0)$  we write

$$f(t) = f_0 + \delta f(t), \quad \lambda(t) = \lambda_0 + \delta \lambda(t), \quad \mu(t) = \mu_0 + \delta \mu(t),$$

substitute this into the system, use the fact that  $(f_0, \lambda_0, \mu_0)$  is a solution, and drop all terms beyond the linear ones in  $(\delta f, \delta \lambda, \delta \mu)$ . At the moment it

may not be obvious that this notation is consistent with the one from the previous sections, because the  $\delta$  terms now have a different meaning. But it turns out below that the notation is indeed consistent, and we obtain the following system:

$$\partial_t \delta f + \frac{1}{\gamma} e^{-\lambda_0} \{\delta f, E\} - \left( \gamma \dot{\delta \lambda} w + e^{\mu_0 - \lambda_0} \delta \mu' \langle v \rangle \right) \phi'(E) \frac{e^{\mu_0} w}{\langle v \rangle} = 0, \quad (5.1)$$

$$e^{-2\lambda_0} (r \delta \lambda' - \delta \lambda (2r \lambda'_0 - 1)) = 4\pi \gamma r^2 \delta \rho, \quad (5.2)$$

$$e^{-2\lambda_0} (r \delta \mu' - \delta \lambda (2r \mu'_0 + 1)) = 4\pi \gamma^2 r^2 \delta p, \quad (5.3)$$

where

$$\delta \rho(t, r) = \int \langle v \rangle \delta f(t, x, v) dv, \quad (5.4)$$

$$\delta p(t, r) = \int \frac{w^2}{\langle v \rangle} \delta f(t, x, v) dv. \quad (5.5)$$

Since the condition of a regular center implies that  $\delta \lambda(t, 0) = 0$  the quantity  $\delta \lambda$  is determined by (5.2) as

$$\delta \lambda(t, r) = \gamma e^{2\lambda_0} \frac{4\pi}{r} \int_0^r s^2 \delta \rho(t, s) ds, \quad (5.6)$$

which agrees with the previous definition of  $\delta \lambda$  in (3.3). The question arises whether the linearized version of (2.6) follows from the linearized system stated above; we will make no use of the linearized version of (2.7) to which this question of course applies as well. Indeed,

$$\dot{\delta \lambda} = -4\pi \gamma r e^{\mu_0 + \lambda_0} \delta \mathcal{J}, \quad (5.7)$$

where

$$\delta \mathcal{J}(t, r) = \int w \delta f(t, x, v) dv. \quad (5.8)$$

To see this we observe first that by (5.6),

$$\dot{\delta \lambda}(t, r) = \gamma e^{2\lambda_0} \frac{4\pi}{r} \int_0^r s^2 \partial_t \delta \rho(t, s) ds.$$

By (5.1) and integration by parts,

$$\partial_t \delta \rho = -e^{\mu_0 - \lambda_0} \int v \cdot \partial_x \delta f dv - 2\mu'_0 e^{\mu_0 - \lambda_0} \int w \delta f dv - \dot{\delta \lambda} (\rho_0 + \gamma p_0).$$



If we substitute this into the equation for  $\delta\dot{\lambda}$  and integrate the first term by parts with respect to  $x$  we find that

$$r\delta\dot{\lambda} + 4\pi r^2 \gamma e^{\mu_0 + \lambda_0} \delta J = -4\pi \gamma e^{2\lambda_0} \int_0^r \left( s\delta\dot{\lambda} + 4\pi s^2 \gamma e^{\mu_0 + \lambda_0} \delta J \right) (\rho_0 + \gamma p_0) s \, ds.$$

Since the term in parenthesis vanishes at the origin it vanishes everywhere, which is the assertion.

Since we are restricting our linear stability theory to the class of dynamically accessible perturbations, we limit ourselves to an existence theorem for such solutions. In particular, we show the essential fact that linearly dynamically accessible data retain this property under the flow of the linearized system.

In order to avoid purely technical complications we assume from now on that in addition to the previous assumptions,  $\Phi \in C^2(\mathbb{R})$ . We comment on this assumption in a remark at the end of this section.

**Theorem 5.1** *Let  $\overset{\circ}{h} \in C^2(\mathbb{R}^6)$  be a spherically symmetric function which according to (3.6) generates a linearly dynamically accessible perturbation  $\delta\overset{\circ}{f}$ . Then there exists a unique solution  $\delta f \in C^1([0, \infty[ \times \mathbb{R}^6)$  to the linearized Einstein-Vlasov system (5.1)–(5.5) with  $\delta f(0) = \delta\overset{\circ}{f}$ . Furthermore, there exists  $h \in C^{1,2}([0, \infty[ \times \mathbb{R}^6)$  such that*

$$\delta f(t) = e^{-\lambda_0} \{h(t), f_0\} + 4\pi \gamma^3 r e^{2\mu_0 + \lambda_0} \phi'(E) \frac{w^2}{\langle v \rangle} \int \phi'(E) h \tilde{w} \, d\tilde{v}, \quad (5.9)$$

*i.e.,  $\delta f(t)$  is linearly dynamically accessible. The generating function  $h$  is the unique solution to the transport equation*

$$\partial_t h + \frac{1}{\gamma} e^{-\lambda_0} \{h, E\} + e^{\mu_0} \delta\lambda \frac{w^2}{\langle v \rangle} + \frac{1}{\gamma} E \delta\mu = 0 \quad (5.10)$$

*with initial value  $h(0) = \overset{\circ}{h}$ .*

**Proof.** The proof proceeds as follows. First we establish existence, uniqueness, and regularity of the solution  $h$  to the equation (5.10), where  $\delta\lambda$  and  $\delta\mu$  are defined via the field equations (5.2) and (5.3), with source terms induced by  $\delta f$  as defined in (5.9). Then we prove that  $(\delta f, \delta\lambda, \delta\mu)$  indeed solves the linearized Einstein-Vlasov system.

Equation (5.10) is a first order inhomogeneous transport equation and its characteristics  $s \mapsto (X(s, t, x, v), V(s, t, x, v))$  are defined as the solutions

of

$$\begin{aligned}\dot{x} &= e^{\mu_0(x) - \lambda_0(x)} \frac{v}{\langle v \rangle}, \\ \dot{v} &= -\frac{1}{\gamma} e^{\mu_0(x) - \lambda_0(x)} \nabla \mu_0(x) \langle v \rangle\end{aligned}$$

with initial condition

$$X(t, t, x, v) = x, \quad V(t, t, x, v) = v.$$

The metric coefficients of the steady state can of course be viewed as functions on  $\mathbb{R}^3$ , and as such,  $\lambda_0 \in C^2(\mathbb{R}^3)$  and  $\mu_0 \in C^3(\mathbb{R}^3)$ , cf. [23, 24]. Using the abbreviations  $z = (x, v)$  and  $Z = (X, V)$  we see that  $Z(s, t, \cdot) : \mathbb{R}^6 \rightarrow \mathbb{R}^6$  is a  $C^2$  diffeomorphism. Now assume that  $h$  is a solution to (5.10). Then integration along the characteristics implies that

$$h(t, z) = \mathring{h}(Z(0, t, z)) - \int_0^t \left( e^{\mu_0} \delta \lambda \frac{w^2}{\langle v \rangle} + \frac{1}{\gamma} E \delta \mu \right) (s, Z(s, t, z)) ds. \quad (5.11)$$

We wish to find a solution to this equation, where  $\delta \lambda$  and  $\delta \mu$  are given as the solutions to the field equations (5.2) and (5.3) with source terms induced by  $\delta f$  which in turn is defined via (5.9).

To construct solutions to (5.11), we apply a simple iteration scheme. For any spherically symmetric function  $g \in C^2(\mathbb{R}^6)$  we define the function  $\delta f_g$  by (5.9) with  $h$  replaced by  $g$ . Clearly,  $\delta f_g \in C^1(\mathbb{R}^6)$ , and  $\delta f_g$  is supported in the support of  $f_0$ . The induced source terms  $\delta \rho_g$  and  $\delta p_g$  have the same regularity and are compactly supported as well. The equation (5.6) shows that the induced metric component  $\delta \lambda_g \in C^2(\mathbb{R}^3)$ . Moreover, the formula in Proposition 3.2 shows that  $\delta \lambda_g$  is also compactly supported. Next we can define  $\delta \mu'_g$  by (5.3) and using the boundary condition  $\delta \mu_g(\infty) = 0$  we find that  $\delta \mu_g \in C^2(\mathbb{R}^3)$  is compactly supported as well. Moreover,

$$\|\delta \lambda_g\|_{C_b^2} + \|\delta \mu_g\|_{C_b^2} \leq C \left( \|\delta \rho_g\|_{C_b^1} + \|\delta p_g\|_{C_b^1} \right) \leq C \|\delta f_g\|_{C_b^1} \leq C \|g\|_{C_b^2} \quad (5.12)$$

where the constant depends on the given steady state and the norms extend only over the support of the steady state which is compact. Suppose now that  $g \in C^{1,2}([0, \infty] \times \mathbb{R}^6)$  so that  $g(t) \in C^2(\mathbb{R}^6)$  for  $t \geq 0$ . Motivated by (5.11) we define

$$(Tg)(t, z) := \mathring{h}(Z(0, t, z)) - \int_0^t \left( e^{\mu_0} \delta \lambda_{g(s)} \frac{w^2}{\langle v \rangle} + \frac{1}{\gamma} E \delta \mu_{g(s)} \right) (Z(s, t, z)) ds. \quad (5.13)$$

It is straight forward to see that  $Tg \in C^{1,2}([0, \infty[ \times \mathbb{R}^6)$ , and the linearity of the problem together with (5.12) imply that for  $g_1, g_2 \in C^{1,2}([0, \infty[, \mathbb{R}^6)$  we obtain the estimate

$$\|Tg_1(t) - Tg_2(t)\|_{C_b^2} \leq C \int_0^t \|g_1(s) - g_2(s)\|_{C_b^2} ds. \quad (5.14)$$

Now let

$$h_0(t, z) := \overset{\circ}{h}(z), \quad h_{n+1} := Th_n.$$

The estimate (5.14) implies that  $h_n$  converges  $t$ -locally uniformly to some  $h \in C^{1,2}([0, \infty[ \times \mathbb{R}^6)$ . Using again the estimates in (5.12) and linearity it follows that  $\delta f_{h_n}$  converges  $t$ -locally uniformly to  $\delta f_h \in C^1([0, \infty[ \times \mathbb{R}^6)$  and  $\delta \lambda_{h_n}, \delta \mu_{h_n}$  converge  $t$ -locally uniformly to  $\delta \lambda_h, \delta \mu_h \in C^{1,2}([0, \infty[ \times \mathbb{R}^6)$ . In particular,  $h$  solves (5.11) and hence also (5.10). The uniqueness of the solution is clear.

In order to show that  $(\delta f, \delta \lambda, \delta \mu)$  solves the linearized Einstein-Vlasov system, it only remains to check the Vlasov equation (5.1) as the two field equations (5.2) and (5.3) hold by definition of  $\delta \lambda$  and  $\delta \mu$ . The definition of  $\delta f$  in terms of  $h$  implies that

$$\partial_t \delta f = e^{-\lambda_0} \{\partial_t h, f_0\} + e^{\mu_0} \gamma \delta \lambda \phi'(E) \frac{w^2}{\langle v \rangle}.$$

Comparing this with (5.1) we see that the latter equation becomes equivalent to the relation

$$\begin{aligned} \{\partial_t h, f_0\} &= -\frac{1}{\gamma} \{\delta f, E\} - e^{2\mu_0} \delta \mu' w \phi'(E) \\ &= -\frac{1}{\gamma} \{e^{-\lambda_0} \{h, f_0\}, E\} - \left\{ e^{\mu_0} \delta \lambda \phi'(E) \frac{w^2}{\langle v \rangle}, E \right\} - e^{2\mu_0} \delta \mu' w \phi'(E). \end{aligned}$$

The fact that  $f_0 = \phi(E)$  and  $\{\phi'(E), E\} = 0 = \{E, E\}$  together with the product rule (2.22) imply that this relation is again equivalent to

$$\begin{aligned} \{\partial_t h, f_0\} &= -\frac{1}{\gamma} \{e^{-\lambda_0} \phi'(E) \{h, E\}, E\} - \left\{ e^{\mu_0} \delta \lambda \phi'(E) \frac{w^2}{\langle v \rangle}, E \right\} \\ &\quad - e^{2\mu_0} \delta \mu' w \phi'(E) \\ &= -\frac{1}{\gamma} \phi'(E) \{e^{-\lambda_0} \{h, E\}, E\} - \phi'(E) \left\{ e^{\mu_0} \delta \lambda \frac{w^2}{\langle v \rangle}, E \right\} \\ &\quad - \frac{1}{\gamma} \phi'(E) \{E \delta \mu, E\}, \end{aligned}$$

i.e.

$$\left\{ \partial_t h + \frac{1}{\gamma} e^{-\lambda_0} \{h, E\} + e^{\mu_0} \delta \lambda \frac{w^2}{\langle v \rangle} + \frac{1}{\gamma} E \delta \mu, f_0 \right\} = 0$$

which is implied by (5.10).  $\square$

**Remark.** We emphasize that a global existence and uniqueness result for general smooth data holds as well. The proof follows a simple iteration scheme analogous to the corresponding result for the Vlasov-Poisson case [5]. We also note that the values of  $h$  outside the support of  $f_0$  are actually irrelevant and it would be sufficient to consider the generating function  $h$  as defined only on this support.

The free energy  $\mathcal{A}$  defined in (4.1) is conserved along solutions of the linearized system. This fact, which is clearly important for our stability result, is shown next.

**Proposition 5.2** *Any linearly dynamically accessible solution as constructed in Theorem 5.1 preserves the energy  $\mathcal{A}$ .*

**Remark.** The above assertion is true for any sufficiently smooth solution of the linearized system, provided in particular that the first integral in  $\mathcal{A}(\delta f(t))$  is defined. As the remark at the end of Section 3 shows, this is the case for linearly dynamically accessible solutions as constructed in Theorem 5.1. In the following proof we make use of this structure only to guarantee the existence of the otherwise questionable integrals.

**Proof of Proposition 5.2.** Clearly,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \iint e^{\lambda_0} \frac{\delta f^2}{|\phi'(E)|} dv dx &= \iint e^{\lambda_0} \frac{\delta f}{|\phi'(E)|} \partial_t \delta f dv dx \\ &= \iint \frac{\delta f}{\gamma \phi'(E)} \{ \delta f, E \} dv dx \\ &\quad - \iint \frac{e^{\lambda_0 + \mu_0}}{\langle v \rangle} \delta f \left( \gamma \dot{\delta \lambda} w + e^{\mu_0 - \lambda_0} \delta \mu' \langle v \rangle \right) dv dx \\ &= \iint \frac{\delta f}{\gamma \phi'(E)} \{ \delta f, E \} dv dx \\ &\quad - \int \left( \gamma e^{\lambda_0 + \mu_0} \delta p \dot{\delta \lambda} + e^{2\mu_0} \delta J \delta \mu' \right) dx. \end{aligned}$$

For the first term on the right hand side we use the product rule (2.22) and

the identity (2.24) to conclude that

$$\begin{aligned} \iint \frac{\delta f}{|\phi'(E)|} \{E, \delta f\} dv dx &= -\frac{1}{2} \iint \frac{1}{\phi'(E)} \{E, \delta f^2\} dv dx \\ &= \frac{1}{2} \iint \delta f^2 \left\{ E, \frac{1}{\phi'(E)} \right\} dv dx = 0. \end{aligned}$$

Hence the linearized field equations (5.3) and (5.7) imply that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \iint e^{\lambda_0} \frac{\delta f^2}{|\phi'(E)|} dv dx &= - \int \left( \gamma e^{\lambda_0 + \mu_0} \delta p \delta \dot{\lambda} + e^{2\mu_0} \delta J \delta \mu' \right) dx \\ &= - \int_0^\infty \left( \delta \dot{\lambda} \frac{e^{\mu_0 - \lambda_0}}{\gamma r^2} (r \delta \mu' - \delta \lambda (2r \mu'_0 + 1)) - \frac{e^{\mu_0 - \lambda_0}}{\gamma r} \delta \dot{\lambda} \right) r^2 dr \\ &= \frac{1}{\gamma} \int_0^\infty e^{\mu_0 - \lambda_0} (2r \mu'_0 + 1) \delta \dot{\lambda} \delta \lambda dr \\ &= \frac{1}{2\gamma} \frac{d}{dt} \int_0^\infty e^{\mu_0 - \lambda_0} (2r \mu'_0 + 1) (\delta \lambda)^2 dr \end{aligned}$$

as required, and the proof is complete.  $\square$

**Remark.** In Theorem 5.1, the assumption  $\Phi \in C^2(\mathbb{R})$  can be relaxed to a  $C^1$  assumption. The existence theory can then be developed for  $h \in C^1([0, \infty \times \mathbb{R}^6])$ . Since  $\delta f$  is then only continuous, it becomes more technical to justify Proposition 5.2, but this can be done analogously to the proof of [5, Theorem 5.1].

## 6 Linear stability

Our coercivity estimate in the form of Corollary 4.5 and the conservation of the free energy according to Proposition 5.2 immediately imply the following result.

**Theorem 6.1** *Let  $\Phi$  satisfy the assumptions stated above, let  $\gamma^*$  and  $C^*$  be as in Theorem 4.2, and let  $0 < \gamma \leq \gamma^*$ . Then the corresponding steady state introduced in Proposition 4.1 is linearly stable in the following sense: For any spherically symmetric function  $\mathring{h} \in C^2(\mathbb{R}^6)$  the solution of the linearized Einstein-Vlasov system (5.1)–(5.5) with the dynamically accessible state  $\delta \mathring{f}$  generated by  $\mathring{h}$  according to (3.6) as initial datum satisfies for all times  $t \geq 0$*

the estimate

$$C^* \iint |\phi'(E)| \left( (rw)^2 \left| \left\{ E, \frac{h_-(t)}{rw} \right\} \right|^2 + \gamma^2 |h_-(t)|^2 \right) dv dx \\ + \frac{1}{2} \iint e^{-\lambda_0} |\phi'(E)| |\{E, h_+(t)\}|^2 dv dx \leq \mathcal{A}(\delta^\circ f).$$

**Remark.** The left hand side is an acceptable measure of the size of the linearized perturbation from the given steady state, in particular, if it vanishes then so does the perturbation  $\delta f(t)$ . The fact that the left hand side controls  $h(t)$  only on the support of the steady state is natural in a linearized approach. The right hand side can clearly be made as small as desired by making the initial perturbation small in a suitable sense.

The result above is acceptable as a linear stability result, except for the fact that  $\gamma = 1/c^2$  has to be chosen small when in a given set of units this quantity has a definite value. We therefore recast our result into one on the stability of a one-parameter family of steady states of the Einstein-Vlasov system with  $\gamma = 1$  by using the scaling properties of the system. This will also provide a quantitative version of the Ze'ldovitch observation which was explained in the introduction.

In order to do so we now denote the Einstein-Vlasov system (2.3)–(2.11), i.e., including the parameter  $\gamma$ , by  $(EV_\gamma)$  and by  $(EV)$  the system with  $\gamma = 1$ . A straight forward computation shows that if  $f$  is a solution of  $(EV_\gamma)$  with the corresponding metric coefficients  $\lambda$  and  $\mu$ , then

$$T^\gamma f(t, x, v) := \gamma^{-3/2} f(t, \gamma^{-1/2} x, \gamma^{-1/2} v)$$

defines a solution of  $(EV)$  with metric coefficients

$$T^\gamma \lambda(t, r) := \lambda(t, \gamma^{-1/2} r), \quad T^\gamma \mu(t, r) := \mu(t, \gamma^{-1/2} r).$$

For a fixed  $\phi$  as specified above, the steady state  $(f_0, \lambda_0, \gamma_0)$  of  $(EV_\gamma)$  with  $0 < \gamma \leq \gamma_0$ —this is actually a family of steady states, one for each  $(EV_\gamma)$ —is now mapped into the one-parameter family of steady states of  $(EV)$  given by

$$]0, \gamma_0] \ni \gamma \mapsto (f_0^\gamma, \lambda_0^\gamma, \mu_0^\gamma) = (T^\gamma f_0, T^\gamma \lambda_0, T^\gamma \mu_0);$$

here  $\gamma_0$  is from Proposition 4.1. It should be carefully noted that all the members of this family are steady states of  $(EV)$ , i.e., of the system with  $\gamma = 1$ , and  $\gamma \in ]0, \gamma_0]$  is now the parameter which parametrizes this family. In particular

$$f_0^\gamma = \gamma^{-3/2} \phi(E) = \phi_\gamma(E)$$

where

$$E(x, v) := e^{\mu_0^\gamma(r)} \sqrt{1 + |v|^2} \text{ and } \phi_\gamma := \gamma^{-3/2} \phi.$$

In order to understand what it means to move to smaller values of  $\gamma$  in this family we observe that by construction  $\mu_0^\gamma(R^\gamma) = 0$  holds for a unique radius  $R^\gamma > 0$  which determines the boundary of the spatial support of the steady state, i.e., the boundary or surface of the galaxy or globular cluster. At the center  $r = 0$  we have  $\mu_0^\gamma(0) = \gamma \mathring{\nu}$ . The expression

$$z := \frac{e^{\mu_0^\gamma(R^\gamma)}}{e^{\mu_0^\gamma(0)}} - 1 = \frac{1}{e^{\gamma \mathring{\nu}}} - 1 > 0$$

is the redshift of a photon which is emitted at the center and received at the surface of the mass distribution, and it is a measure for how strong relativistic effects in the configuration are. Expressed in units where the speed of light  $c = 1$  the above limit  $\gamma \rightarrow 0$  means that we consider steady states for which  $z$  is close to 0, i.e., relativistic effects are weak.

In order to recast our stability result for the case of (EV) we have to check how the various quantities which are involved behave under the scaling operator  $T^\gamma$ . To this end we rename the particle energy in the context of (EV $_\gamma$ ) as

$$E_\gamma = E_\gamma(x, v) = e^{\mu_0(r)} \sqrt{1 + \gamma |v|^2} = E(\gamma^{1/2} x, \gamma^{1/2} v).$$

In order to understand the behavior of the Poisson bracket (2.21) under the scaling operator  $T^\gamma$ , we define for a given function  $h = h(x, v)$ ,

$$h^\gamma(x, v) := \gamma^{-1} h(\gamma^{1/2} x, \gamma^{1/2} v).$$

Then the relation

$$\{h, f_0^\gamma\}(x, v) = T^\gamma \{h^\gamma, f_0\}$$

holds. Next, a simple change of variables argument implies that

$$\begin{aligned} & 4\pi r e^{\mu_0^\gamma + \lambda_0^\gamma} \phi_\gamma'(E) \frac{w^2}{\langle v \rangle} \int \phi_\gamma'(E) h \tilde{w} d\tilde{v} \\ &= T^\gamma \left( 4\pi \gamma^3 r e^{\mu_0 + \lambda_0} \phi'(E_\gamma) \frac{w^2}{\langle v \rangle} \int \phi'(E_\gamma) h^\gamma \tilde{w} d\tilde{v} \right). \end{aligned}$$

In particular, if

$$\delta f_h = e^{-\lambda_0^\gamma} \{h, f_0^\gamma\} + 4\pi r e^{\mu_0^\gamma + \lambda_0^\gamma} \phi_\gamma'(E) \frac{w^2}{\langle v \rangle} \int \phi_\gamma'(E) h \tilde{w} d\tilde{v} \quad (6.1)$$

is the linearly dynamically accessible state for (EV) generated by  $h$ , then

$$\delta f_h = T^\gamma \delta f_{h^\gamma}, \quad (6.2)$$

where  $\delta f_{h^\gamma}$  is the associated linearly dynamically accessible state for  $(\text{EV}_\gamma)$ , generated by  $h^\gamma$  as defined in (3.6).

Finally, to understand the behavior of the free energy we first must again be careful with the notation. The free energy associated with the steady state  $(f_0, \lambda_0, \mu_0)$  of  $(\text{EV}_\gamma)$  and defined in (4.1) is still denoted by  $\mathcal{A}$ , while the corresponding quantity associated with the steady state  $(f_0^\gamma, \lambda_0^\gamma, \mu_0^\gamma)$  of (EV) is defined by

$$\mathcal{A}^\gamma(\delta f) := \frac{1}{2} \iint \frac{e^{\lambda_0^\gamma}}{|\phi'_\gamma(E)|} (\delta f)^2 dv dx - \frac{1}{2} \int_0^\infty e^{\mu_0^\gamma - \lambda_0^\gamma} \left( 2r (\mu_0^\gamma)' + 1 \right) (\delta \lambda)^2 dr.$$

A simple calculation shows that

$$\mathcal{A}(\delta f) = \gamma^{-3/2} \mathcal{A}^\gamma(T^\gamma \delta f). \quad (6.3)$$

We now recast Theorem 6.1, which is a result for  $(\text{EV}_\gamma)$ , into a result for (EV).

**Theorem 6.2** *Let  $\Phi$  satisfy the assumptions stated above and let  $\gamma^*$  and  $C^*$  be as in Theorem 4.2. Then provided  $0 < \gamma \leq \gamma^*$  the corresponding steady state  $(f_0^\gamma, \lambda_0^\gamma, \mu_0^\gamma)$  of (EV) is linearly stable in the following sense: For any spherically symmetric function  $\mathring{h} \in C^2(\mathbb{R}^6)$  the solution  $f_h$  of the linearized Einstein-Vlasov system with the dynamically accessible state  $\delta \mathring{f}$  generated by  $\mathring{h}$  according to (6.1) as initial datum satisfies for all times  $t \geq 0$  the estimate*

$$\begin{aligned} C^* \iint |\phi'_\gamma(E)| \left( (rw)^2 \left| \left\{ E, \frac{h_-(t)}{rw} \right\} \right|^2 + |h_-(t)|^2 \right) dv dx \\ + \frac{1}{2} \iint e^{-\lambda_0^\gamma} |\phi'_\gamma(E)| |\{E, h_+(t)\}|^2 dv dx \leq \mathcal{A}^\gamma(\delta \mathring{f}). \end{aligned}$$

**Proof.** In the statement of the theorem the linearized Einstein-Vlasov system is now the system (5.1)–(5.5) with  $\gamma = 1$ , and  $\mathcal{A}^\gamma$  is a conserved quantity, in particular along solutions which are linearly dynamically accessible from the steady state  $(f_0^\gamma, \lambda_0^\gamma, \mu_0^\gamma)$  about which the system (EV) was linearized. Hence by (6.2) and (6.3),

$$\mathcal{A}^\gamma(\delta \mathring{f}) = \mathcal{A}^\gamma(\delta f_h(t)) = \mathcal{A}^\gamma(T^\gamma \delta f_{h^\gamma}(t)) = \gamma^{3/2} \mathcal{A}(\delta f_{h^\gamma}(t)).$$



If we apply Corollary 4.5 it follows that

$$\begin{aligned} \mathcal{A}^\gamma(\delta^\circ f) &\geq C^* \gamma^{3/2} \iint |\phi'(E_\gamma)| \left( (rw)^2 \left| \left\{ E_\gamma, \frac{h_-^\gamma(t)}{rw} \right\} \right|^2 + \gamma^2 |h_-^\gamma(t)|^2 \right) dv dx \\ &\quad + \gamma^{3/2} \frac{1}{2} \iint e^{-\lambda_0} |\phi'(E_\gamma)| \left| \{E_\gamma, h_+^\gamma(t)\} \right|^2 dv dx. \end{aligned}$$

Now we apply a change of variables to turn  $h^\gamma$  into  $h$  under these integrals and the result follows.  $\square$

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